

# A CANONICAL QUADRATIC FORM ON THE DETERMINANT LINE OF A FLAT VECTOR BUNDLE

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**ABSTRACT.** We introduce and study a canonical quadratic form, called the torsion quadratic form, of the determinant line of a flat vector bundle over a closed oriented odd-dimensional manifold. This quadratic form carries less information than the refined analytic torsion, introduced in our previous work, but is easier to construct and closer related to the combinatorial Farber-Turaev torsion. In fact, the torsion quadratic form can be viewed as an analytic analogue of the Poincaré-Reidemeister scalar product, introduced by Farber and Turaev. Moreover, it is also closely related to the complex analytic torsion defined by Cappell and Miller and we establish the precise relationship between the two. In addition, we show that up to an explicit factor, which depends on the Euler structure, and a sign the Burghelea-Haller complex analytic torsion, whenever it is defined, is equal to our quadratic form. We conjecture a formula for the value of the torsion quadratic form at the Farber-Turaev torsion and prove some weak version of this conjecture. As an application we establish a relationship between the Cappell-Miller and the combinatorial torsions.

## 1. INTRODUCTION

In [5], we constructed a new invariant of a flat vector bundle  $(E, \nabla)$  over a closed oriented manifold  $M$  of odd dimension  $d = 2r - 1$ . It is a quadratic form  $\tau = \tau_\nabla$ , called the *torsion quadratic form*, on the determinant line  $\text{Det}(H^\bullet(M, E))$  of the cohomology of  $E$ , which we defined in terms of another, more sophisticated invariant, the refined analytic torsion  $\rho_{\text{an}} \in \text{Det}(H^\bullet(M, E))$ , constructed in [6, 8, 7].

The invariant  $\tau$  is closely related to the quadratic form  $\tau^{\text{BH}} = \tau_{\nabla, b}^{\text{BH}}$ , introduced by Burghelea and Haller [11]. To construct  $\tau^{\text{BH}}$  they need to require that the bundle  $E$  admits a complex valued non-degenerate bilinear form  $b$ . The definition of  $\tau^{\text{BH}}$  is similar to the definition of the Ray-Singer torsion, but instead of the standard Laplacians on differential forms uses the non-self-adjoint Laplace-type operators  $\Delta_b = \nabla \nabla_b^\# + \nabla_b^\# \nabla$ , where  $\nabla_b^\#$  denotes the adjoint of  $\nabla$  with respect to the bilinear form  $b$ . Recall that the Ray-Singer torsion is a combination of the square roots of the determinants of the standard Laplacians. Since the determinants of the non-self-adjoint operators  $\Delta_b$  are complex numbers their square roots are not canonically defined. This is the reason why Burghelea and Haller defined  $\tau^{\text{BH}}$  in terms of the determinants of  $\Delta_b$  rather than their square roots, extending in this way the square of the Ray-Singer torsion.

Farber and Turaev [18, 19] defined a combinatorial torsion  $\rho_{\varepsilon, \mathfrak{o}} \in \text{Det}(H^\bullet(M, E))$  which depends on the orientation  $\mathfrak{o}$  of the cohomology  $H^\bullet(M)$  and on the Euler structure  $\varepsilon$  introduced by Turaev [27, 28]. It was noticed by Burghelea [9] that the Euler structure  $\varepsilon$  can be described by a closed form  $\alpha_\varepsilon \in \Omega^{d-1}(M)$ . Extending the classical Ray-Singer conjecture, [24, 15, 23, 4], Burghelea and Haller conjectured that

$$\tau_{\nabla, b}^{\text{BH}}(\rho_{\varepsilon, \mathfrak{o}}) = e^{\int_M \omega_{\nabla, b} \wedge \alpha_\varepsilon}, \quad (1.1)$$

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where  $\omega_{\nabla, b} = -\frac{1}{2} \text{Tr } b^{-1} \nabla b$  is the Kamber-Tondeur form, which measures the non-flatness of the bilinear form  $b$ . This conjecture was proven independently by Burghlelea-Haller [10] and Su-Zhang [26].

In [5], we showed that  $\tau^{\text{BH}} = \pm \tau$  whenever  $\tau^{\text{BH}}$  is defined and extended the Burghlelea-Haller conjecture to the case when  $\tau^{\text{BH}}$  is not defined. More precisely, we conjectured, cf. [5, Conjecture 1.12], that

$$\tau_{\nabla}(\rho_{\varepsilon, \mathbf{o}}) = e^{2\pi i \langle \mathbf{Arg}_{\nabla}, c(\varepsilon) \rangle}. \quad (1.2)$$

Here  $c(\varepsilon) \in H_1(M, \mathbb{Z})$  is the characteristic class of the Euler structure  $\varepsilon$ , cf. [28, §5.3];  $\mathbf{Arg}_{\nabla} \in H^1(M, \mathbb{C}/\mathbb{Z})$  is the unique cohomology class such that for every closed curve  $\gamma$  in  $M$  we have

$$\det(\text{Mon}_{\nabla}(\gamma)) = \exp(2\pi i \langle \mathbf{Arg}_{\nabla}, [\gamma] \rangle),$$

where  $\text{Mon}_{\nabla}(\gamma)$  denotes the monodromy of the flat connection  $\nabla$  along the curve  $\gamma$ ; finally,  $\langle \cdot, \cdot \rangle$  denotes the natural pairing  $H^1(M, \mathbb{C}/\mathbb{Z}) \times H_1(M, \mathbb{Z}) \rightarrow \mathbb{C}/\mathbb{Z}$ .

Note that (1.1) implies (1.2) whenever  $\tau^{\text{BH}}$  is defined, see [5, §1.11]. In [5] we proved the following weak version of Conjecture (1.2): For each connected component  $\mathcal{C}$  of the space of flat connections on  $E$  there exists a constant  $R_{\mathcal{C}} \in \mathbb{C}$  with  $|R_{\mathcal{C}}| = 1$ , such that

$$\tau_{\nabla}(\rho_{\varepsilon, \mathbf{o}}) = R_{\mathcal{C}} \cdot e^{2\pi i \langle \mathbf{Arg}_{\nabla}, c(\varepsilon) \rangle}. \quad (1.3)$$

Farber and Turaev [19, §9] introduced a bilinear form  $\langle \cdot, \cdot \rangle_{\text{PR}}$  on  $\text{Det}(H^{\bullet}(M, E))$ , which they call the cohomological Poincaré-Reidemeister scalar product. This is an invariant which refines the Poincaré-Reidemeister metric introduced by Farber [16]. It follows from Theorem 9.4 of [19] that Conjecture (1.2) is equivalent to the statement that

$$\tau_{\nabla}(\cdot) = (-1)^z \langle \cdot, \cdot \rangle_{\text{PR}},$$

where  $z \in \mathbb{N}$  is defined in formula (6.5) of [19].

Another related invariant  $T \in \text{Det}(H^{\bullet}(M, E)) \otimes \text{Det}(H^{\bullet}(M, E))$  was introduced by Cappell and Miller [14]. To define  $T$  they also used non-self-adjoint Laplace-type operators, but different from the ones used by Burghlelea and Haller. In fact, they consider the square  $\mathcal{B}^2$  of the Atiyah-Patodi-Singer odd signature operator  $\mathcal{B} = \mathcal{B}(\nabla, g^M)$  and, hence, don't need any additional assumptions on  $E$ . Further in [14], Cappell and Miller conjectured that, in an appropriate sense, their torsion is equal to the Reidemeister torsion of the bundle  $E \oplus E^*$ , where  $E^*$  denotes the dual bundle to  $E$ .

The goal of this paper is to present a simple construction of the torsion quadratic form  $\tau$ , implicitly already contained in [8]. We collect only those parts of [6, 8, 5], which are needed for this purpose. In particular, we bypass the refined analytic torsion. Recall that the definition of the refined analytic torsion in [6, 8] uses the graded determinant of the odd signature operator  $\mathcal{B}$ , leading to a rather complicated analysis, involving the determinant of  $\mathcal{B}^2$  and the  $\eta$ -invariant. In contrast, the definition of  $\tau$  only involves the determinant of the Laplace-type operator  $\mathcal{B}^2$ . It turns out that the construction of  $T$  by Cappell and Miller is very similar to our construction of  $\tau$ , as it uses the same operator  $\mathcal{B}^2$ . We establish the precise relationship of  $T$  with  $\tau$ . It turns out that  $T$  is the dual of  $\tau$ . As an application we prove a weak version of the Cappell-Miller conjecture.

## 2. THE QUADRATIC FORM ON THE DETERMINANT LINE OF A FINITE DIMENSIONAL COMPLEX

In this section we define a canonical quadratic form on a finite dimensional complex with involution.

**2.1. The construction of a quadratic form.** Let

$$(C^\bullet, \partial) : 0 \rightarrow C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \dots \xrightarrow{\partial} C^d \rightarrow 0 \quad (2.1)$$

be a complex of *finite dimensional* complex vector spaces of odd length  $d = 2r - 1$ . A *chirality operator*  $\Gamma : C^\bullet \rightarrow C^\bullet$  is an involution such that  $\Gamma(C^j) = C^{d-j}$ , for all  $j = 0, \dots, d$ . Consider the determinant line

$$\text{Det}(C^\bullet) := \bigotimes_{j=0}^d \text{Det}(C^j)^{(-1)^j},$$

where  $\text{Det}(C^j)^{-1} := \text{Hom}(\text{Det}(C^j), \mathbb{C})$  denotes the dual of  $C^j$ . For an element  $c_j \in \text{Det}(C^j)$  we denote by  $c_j^{-1}$  the unique element in  $\text{Det}(C^j)^{-1}$  satisfying  $c_j^{-1}(c_j) = 1$ . We also denote by  $\Gamma c_j \in \text{Det}(C^{d-j})$  the image of  $c_j$  under the map  $\text{Det}(C^j) \rightarrow \text{Det}(C^{d-j})$  induced by  $\Gamma : C^j \rightarrow C^{d-j}$ .

Denote by  $H^\bullet(\partial)$  the cohomology of the complex  $(C^\bullet, \partial)$ . Let

$$\phi_{C^\bullet} : \text{Det}(C^\bullet) \rightarrow \text{Det}(H^\bullet(\partial)) \quad (2.2)$$

be the canonical isomorphism, cf. [22].<sup>1</sup>

Note that any element  $c \in \text{Det}(C^\bullet)$  can be written in a form  $c = c_0 \otimes c_1^{-1} \otimes \dots \otimes c_d^{-1}$ , where  $c_j \in \text{Det}(C^j)$ . Hence, any element of  $\text{Det}(H^\bullet(\partial))$  can be written as  $\phi_{C^\bullet}(c_0 \otimes c_1^{-1} \otimes \dots \otimes c_d^{-1})$ .

**Definition 2.2.** The torsion quadratic form  $\tau_r$  of the pair  $(C^\bullet, \Gamma)$  is the unique quadratic form on  $\text{Det}(H^\bullet(\partial))$  such that

$$\tau_r(\phi_{C^\bullet}(c_0 \otimes c_1^{-1} \otimes \dots \otimes c_d^{-1})) = \prod_{j=0}^d \left[ c_j^{-1}(\Gamma c_{d-j}) \right]^{(-1)^{j+1}}. \quad (2.3)$$

**2.3. Relationship with the refined torsion.** In [8] we introduced a canonical element of  $\text{Det}(H^\bullet(\partial))$ , called the *refined torsion* of the pair  $(C^\bullet, \Gamma)$ , as follows. For each  $j = 0, \dots, r-1$ , fix an element  $c_j \in \text{Det}(C^j)$  and set

$$c_r := (-1)^{\mathcal{R}(C^\bullet)} \cdot c_0 \otimes c_1^{-1} \otimes \dots \otimes c_{r-1}^{(-1)^{r-1}} \otimes (\Gamma c_{r-1})^{(-1)^r} \otimes (\Gamma c_{r-2})^{(-1)^{r-1}} \otimes \dots \otimes (\Gamma c_0)^{-1} \quad (2.4)$$

of  $\text{Det}(C^\bullet)$ , where

$$\mathcal{R}(C^\bullet) = \frac{1}{2} \sum_{j=0}^{r-1} \dim C^j \cdot (\dim C^j - 1). \quad (2.5)$$

It is easy to see that  $c_r$  is independent of the choice of  $c_0, \dots, c_{r-1}$ . The *refined torsion* of the pair  $(C^\bullet, \Gamma)$  is the element

$$\rho_r = \rho_{C^\bullet, \Gamma} := \phi_{C^\bullet}(c_r) \in \text{Det}(H^\bullet(\partial)). \quad (2.6)$$

It follows immediately from (2.3) and (2.6) that

$$\tau_r(\rho_r) = 1. \quad (2.7)$$

**2.4. An acyclic complex.** Suppose the complex  $(C^\bullet, \partial)$  is acyclic. Then  $\text{Det}(H^\bullet(\partial))$  is naturally isomorphic to  $\mathbb{C}$ . Using this isomorphism we identify  $\tau_r$  with the complex number

$$\hat{\tau}_r := \tau_r(1) \in \mathbb{C} \setminus \{0\}, \quad 1 \in \mathbb{C} \simeq \text{Det}(H^\bullet(\partial)). \quad (2.8)$$

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<sup>1</sup>In [8] we used a sign refined version of this isomorphism, but we don't need this more complicated version in the present paper.

**2.5. Calculation of the refined torsion of a finite dimensional complex.** To compute the refined torsion we introduce the operator

$$\mathcal{B} := \Gamma \partial + \partial \Gamma. \quad (2.9)$$

This operator is a finite dimensional analogue of the signature operator on an odd-dimensional manifold, see [1, p. 44], [2, p. 405], [20, pp. 64–65], and Section 3 of this paper. Then

$$\mathcal{B}^2 = \Gamma \partial \Gamma \partial + \partial \Gamma \partial \Gamma. \quad (2.10)$$

*Remark 2.6.* In many interesting applications, cf. Section 3, there exists a scalar product on  $C^\bullet$  such that the adjoint of  $\partial$  satisfies  $\partial^* = \Gamma \partial \Gamma$ . Then  $\mathcal{B}^2$  is equal to the Laplacian of the complex  $C^\bullet$ .

Let us first treat the case where the signature operator  $\mathcal{B}$  is bijective.

**Lemma 2.7.** *Suppose that the operator  $\mathcal{B}$  is invertible. Then the complex  $(C^\bullet, \partial)$  is acyclic and the complex number  $\hat{\tau}_\Gamma$ , cf. (2.8), is given by*

$$\hat{\tau}_\Gamma = \prod_{j=0}^d \text{Det}(\mathcal{B}^2|_{C^j})^{(-1)^j j}. \quad (2.11)$$

*Proof.* Since  $\Gamma^2 = \text{Id}$ , for every  $a \in \text{Det}(C^\bullet)$ ,  $b \in \text{Det}(C^{d-\bullet})$ , we have

$$a^{-1}(\Gamma b) = (\Gamma a)^{-1}(b) = \frac{1}{b^{-1}(\Gamma a)}.$$

Hence, for all  $j = 0, \dots, d$ ,

$$[c_j^{-1}(\Gamma c_{d-j})]^{(-1)^{j+1}} = [c_{d-j}^{-1}(\Gamma c_j)]^{(-1)^{d-j+1}}$$

and the definition (2.3) of  $\tau_\Gamma$  can be rewritten as

$$\tau_\Gamma(\phi_{C^\bullet}(c_0 \otimes c_1^{-1} \otimes \dots \otimes c_d^{-1})) = \left[ \prod_{j=0}^{r-1} [c_j^{-1}(\Gamma c_{d-j})]^{(-1)^{j+1}} \right]^2. \quad (2.12)$$

As, by assumption, the operator  $\mathcal{B} = \Gamma \partial + \partial \Gamma$  is invertible, for each  $j = 0, \dots, n$  we have a direct sum decomposition

$$C^j = A^j \oplus B^j,$$

where  $A^j = \text{Ker}(\partial|_{C^j})$  and  $B^j = \Gamma \partial(C^{d-j-1})$ . It follows that the complex  $(C^\bullet, \partial)$  is acyclic and  $A^j = \partial(B^{j-1})$  for all  $j = 1, \dots, d$ . Set  $n_j = \dim B^j$ . Then  $n_j = n_{d-j-1}$  and  $\dim A^j = n_{j-1}$ .

For  $j = 0, \dots, r-1$  choose a basis  $\{b_1^j, \dots, b_{n_j}^j\}$  of  $B^j$ . For  $j = r, \dots, d-1$  set  $b_i^j = \Gamma \partial b_i^{d-j-1}$ . Then for any  $j = 0, \dots, d-1$ ,  $\{b_1^j, \dots, b_{n_j}^j\}$  is a basis of  $B^j$ . It follows that  $\{\partial b_1^{j-1}, \dots, \partial b_{n_{j-1}}^{j-1}\}$  is a basis of  $A^j$ , for  $j = 1, \dots, d$ . Hence,

$$\{\partial b_1^{j-1}, \dots, \partial b_{n_{j-1}}^{j-1}, b_1^j, \dots, b_{n_j}^j\}$$

is a basis of  $C^j$  ( $j = 1, \dots, d-1$ ),  $\{b_1^0, \dots, b_{n_0}^0\}$  is the basis of  $C^0$ , and  $\{\partial b_1^{d-1}, \dots, \partial b_{n_{d-1}}^{d-1}\}$  is the basis of  $C^d$ . Set

$$c_0 = b_1^0 \wedge \dots \wedge b_{n_0}^0, \quad c_d = \partial b_1^{d-1} \wedge \dots \wedge \partial b_{n_{d-1}}^{d-1},$$

and, for  $j = 1, \dots, d-1$ ,

$$c_j = \partial b_1^{j-1} \wedge \dots \wedge \partial b_{n_{j-1}}^{j-1} \wedge b_1^j \wedge \dots \wedge b_{n_j}^j \in \text{Det}(C^j).$$

By the definition of the map  $\phi_{C^\bullet} : \text{Det}(C^\bullet) \rightarrow \text{Det}(H^\bullet(\partial)) \simeq \mathbb{C}$

$$\phi_{C^\bullet}(c_0 \otimes c_1^{-1} \otimes \dots \otimes c_d^{-1}) = 1 \in \mathbb{C}.$$

Therefore, by (2.8) and (2.12),

$$\hat{\tau}_r = \left[ \prod_{j=0}^{r-1} \left[ c_j^{-1}(\Gamma c_{d-j}) \right]^{(-1)^{j+1}} \right]^2. \quad (2.13)$$

We now need to compute the numbers  $c_j^{-1}(\Gamma c_{d-j})$ . Assume first, that  $j = 1, \dots, r-2$ . Then  $c_j^{-1}(\Gamma c_{d-j})$  is equal to the determinant of the operator  $S_j : C^j \rightarrow C^j$ , which transforms the basis  $\{\partial b_1^{j-1}, \dots, \partial b_{n_{j-1}}^{j-1}, b_1^j, \dots, b_{n_j}^j\}$  to the basis

$$\{\Gamma \partial b_1^{d-j-1}, \dots, \Gamma \partial b_{n_{d-j-1}}^{d-j-1}, \Gamma b_1^{d-j}, \dots, \Gamma b_{n_{d-j}}^{d-j}\} = \{\Gamma \partial \Gamma \partial b_1^j, \dots, \Gamma \partial \Gamma \partial b_{n_j}^j, \partial b_1^{j-1}, \dots, \partial b_{n_{j-1}}^{j-1}\}.$$

Here we used that, by construction,  $\Gamma \partial b_i^j = b_i^{d-j-1}$ , for any  $i = 1, \dots, n_j$  and  $b_i^{d-j} = \Gamma \partial b_i^{j-1}$  for any  $i = 1, \dots, n_{j-1}$ . We conclude that

$$c_j^{-1}(\Gamma c_{d-j}) = \text{Det}(S_j) = \pm \text{Det}(\Gamma \partial \Gamma \partial|_{B^j}), \quad j = 1, \dots, r-2. \quad (2.14)$$

Similarly,  $c_0^{-1}(\Gamma c_d)$  is the determinant of the operator which transforms the basis  $\{b_1^0, \dots, b_{n_0}^0\}$  to the basis

$$\{\Gamma \partial b_1^{d-1}, \dots, \Gamma \partial b_{n_{d-1}}^{d-1}\} = \{\Gamma \partial \Gamma \partial b_1^0, \dots, \Gamma \partial \Gamma \partial b_{n_0}^0\}.$$

Thus,

$$c_0^{-1}(\Gamma c_d) = \text{Det}(\Gamma \partial \Gamma \partial|_{B^0}). \quad (2.15)$$

Finally,  $c_{r-1}^{-1}(\Gamma c_r)$  is equal to the determinant of the operator which transforms the basis  $\{\partial b_1^{r-2}, \dots, \partial b_{n_{r-2}}^{r-2}, b_1^{r-1}, \dots, b_{n_{r-1}}^{r-1}\}$  to the basis

$$\{\Gamma \partial b_1^{r-1}, \dots, \Gamma \partial b_{n_{r-1}}^{r-1}, \Gamma b_1^r, \dots, \Gamma b_{n_r}^r\} = \{\Gamma \partial b_1^{r-1}, \dots, \Gamma \partial b_{n_{r-1}}^{r-1}, \partial b_1^{r-2}, \dots, \partial b_{n_{r-2}}^{r-2}\},$$

and, hence, is equal to  $\pm \text{Det}(\Gamma \partial|_{B^{r-1}})$ . Therefore,

$$[c_{r-1}^{-1}(\Gamma c_r)]^2 = \text{Det}(\Gamma \partial|_{B^{r-1}})^2 = \text{Det}(\Gamma \partial \Gamma \partial|_{B^{r-1}}). \quad (2.16)$$

Combining equations (2.13)–(2.16) we obtain

$$\hat{\tau}_r = \left[ \prod_{j=0}^{r-2} \left[ \text{Det}(\Gamma \partial \Gamma \partial|_{B^j}) \right]^{(-1)^{j+1}} \right]^2 \cdot \text{Det}(\Gamma \partial \Gamma \partial|_{B^{r-1}}). \quad (2.17)$$

The isomorphism  $\Gamma \partial : B^j \rightarrow B^{d-j-1}$  intertwines the operators  $\Gamma \partial \Gamma \partial|_{B^j}$  and  $\Gamma \partial \Gamma \partial|_{B^{d-j-1}}$ . Hence,

$$\text{Det}(\Gamma \partial \Gamma \partial|_{B^j}) = \text{Det}(\Gamma \partial \Gamma \partial|_{B^{d-j-1}})$$

and (2.17) can be rewritten as

$$\hat{\tau}_r = \prod_{j=0}^{d-1} \left[ \text{Det}(\Gamma \partial \Gamma \partial|_{B^j}) \right]^{(-1)^{j+1}}. \quad (2.18)$$

The isomorphism  $\partial : B^{j-1} \rightarrow A^j$  intertwines the operators  $\Gamma \partial \Gamma \partial|_{B^{j-1}}$  and  $\partial \Gamma \partial \Gamma|_{A^j}$ . Hence,

$$\text{Det}(\Gamma \partial \Gamma \partial|_{B^{j-1}}) = \text{Det}(\partial \Gamma \partial \Gamma|_{A^j}), \quad j = 1, \dots, d.$$

Thus, from (2.10), we conclude that

$$\text{Det}(\mathcal{B}^2|_{C^0}) = \text{Det}(\Gamma \partial \Gamma \partial|_{B^0}), \quad \text{Det}(\mathcal{B}^2|_{C^d}) = \text{Det}(\Gamma \partial \Gamma \partial|_{B^{d-1}}).$$

and, for  $j = 1, \dots, d-1$ ,

$$\text{Det}(\mathcal{B}^2|_{C^j}) = \text{Det}(\Gamma \partial \Gamma \partial|_{B^j}) \cdot \text{Det}(\partial \Gamma \partial \Gamma|_{A^j}) = \text{Det}(\Gamma \partial \Gamma \partial|_{B^j}) \cdot \text{Det}(\Gamma \partial \Gamma \partial|_{B^{j-1}})$$

Therefore,

$$\begin{aligned} \prod_{j=0}^d \text{Det}(\mathcal{B}^2|_{C^j})^{(-1)^j j} &= \prod_{j=0}^{d-1} \text{Det}(\text{Det}(\Gamma \partial \Gamma \partial|_{B^j}))^{(-1)^j j} \cdot \prod_{j=1}^d \text{Det}(\text{Det}(\Gamma \partial \Gamma \partial|_{B^{j-1}}))^{(-1)^j j} \\ &= \prod_{j=0}^{d-1} \text{Det}(\text{Det}(\Gamma \partial \Gamma \partial|_{B^j}))^{(-1)^{j+1}}. \end{aligned} \quad (2.19)$$

Combining (2.19) and (2.18) we obtain (2.11).  $\square$

To compute the torsion quadratic form in the case  $\mathcal{B}$  is *not* bijective, note that, for  $j = 0, \dots, d$ , the operator  $\mathcal{B}^2$  maps  $C^j$  into itself. For each  $j = 0, \dots, d$  and an arbitrary interval  $\mathcal{I}$ , denote by  $C_{\mathcal{I}}^j \subset C^j$  the linear span of the generalized eigenvectors of the restriction of  $\mathcal{B}^2$  to  $C^j$ , corresponding to eigenvalues  $\lambda$  with  $|\lambda| \in \mathcal{I}$ . Since both operators,  $\Gamma$  and  $\partial$ , commute with  $\mathcal{B}$  (and, hence, with  $\mathcal{B}^2$ ),  $\Gamma(C_{\mathcal{I}}^j) \subset C_{\mathcal{I}}^{d-j}$  and  $\partial(C_{\mathcal{I}}^j) \subset C_{\mathcal{I}}^{j+1}$ . Hence, we obtain a subcomplex  $C_{\mathcal{I}}^\bullet$  of  $C^\bullet$  and the restriction  $\Gamma_{\mathcal{I}}$  of  $\Gamma$  to  $C_{\mathcal{I}}^\bullet$  is a chirality operator for  $C_{\mathcal{I}}^\bullet$ . We denote by  $H_{\mathcal{I}}^\bullet(\partial)$  the cohomology of the complex  $(C_{\mathcal{I}}^\bullet, \partial_{\mathcal{I}})$ .

Denote by  $\partial_{\mathcal{I}}$  and  $\mathcal{B}_{\mathcal{I}}$  the restrictions of  $\partial$  and  $\mathcal{B}$  to  $C_{\mathcal{I}}^\bullet$ . Then  $B_{\mathcal{I}} = \Gamma_{\mathcal{I}} \partial_{\mathcal{I}} + \partial_{\mathcal{I}} \Gamma_{\mathcal{I}}$  and one easily shows (cf. Lemma 5.8 of [8]) that  $(C_{\mathcal{I}}^\bullet, \partial_{\mathcal{I}})$  is acyclic if  $0 \notin \mathcal{I}$ .

For each  $\lambda \geq 0$ ,  $C^\bullet = C_{[0, \lambda]}^\bullet \oplus C_{(\lambda, \infty)}^\bullet$  and  $H_{(\lambda, \infty)}^\bullet(\partial) = 0$  whereas  $H_{[0, \lambda]}^\bullet(\partial) \simeq H^\bullet(\partial)$ . Hence, there are canonical isomorphisms

$$\Phi_\lambda : \text{Det}(H_{(\lambda, \infty)}^\bullet(\partial)) \longrightarrow \mathbb{C}, \quad \Psi_\lambda : \text{Det}(H_{[0, \lambda]}^\bullet(\partial)) \longrightarrow \text{Det}(H^\bullet(\partial)).$$

In the sequel, we will write  $t$  for  $\Phi_\lambda(t) \in \mathbb{C}$ .

**Lemma 2.8.** *For every  $x \in \text{Det}(H^\bullet(\partial))$  and every  $\lambda \geq 0$  we have*

$$\tau_{\Gamma}(x) = \left[ \prod_{j=0}^d \text{Det}(\mathcal{B}_{(\lambda, \infty)}^2|_{C_{(\lambda, \infty)}^j})^{(-1)^j j} \right] \cdot \tau_{\Gamma_{[0, \lambda]}}(\Psi_\lambda^{-1}(x)). \quad (2.20)$$

*In particular, the right hand side of (2.20) is independent of  $\lambda \geq 0$ .*

*Proof.* For each  $j = 0, \dots, d$  fix  $c'_j \in \text{Det}(C_{[0, \lambda]}^j)$  and  $c''_j \in \text{Det}(C_{(\lambda, \infty)}^j)$ . Then, using the natural isomorphism

$$\text{Det}(C_{[0, \lambda]}^j) \otimes \text{Det}(C_{(\lambda, \infty)}^j) \simeq \text{Det}(C_{[0, \lambda]}^j \oplus C_{(\lambda, \infty)}^j) = \text{Det}(C^j),$$

we can regard the tensor product  $c_j := c'_j \otimes c''_j$  as an element of  $\text{Det}(C^j)$ . Applying (2.3) twice, we obtain

$$\begin{aligned} \tau_{\Gamma}(\phi_{C^\bullet}(c_0 \otimes c_1^{-1} \otimes \dots \otimes c_d^{-1})) &= \prod_{j=0}^d \left[ c_j^{-1}(\Gamma c_{d-j}) \right]^{(-1)^{j+1}} \\ &= \prod_{j=0}^d \left[ (c'_j)^{-1}(\Gamma c'_{d-j}) \right]^{(-1)^{j+1}} \cdot \prod_{j=0}^d \left[ (c''_j)^{-1}(\Gamma c''_{d-j}) \right]^{(-1)^{j+1}} \\ &= \tau_{\Gamma_{[0, \lambda]}}(\phi_{C_{[0, \lambda]}^\bullet}(c'_0 \otimes (c'_1)^{-1} \otimes \dots \otimes (c'_d)^{-1})) \cdot \tau_{\Gamma_{(\lambda, \infty)}}(\phi_{C_{(\lambda, \infty)}^\bullet}(c''_0 \otimes (c''_1)^{-1} \otimes \dots \otimes (c''_d)^{-1})). \end{aligned} \quad (2.21)$$

Let us now choose  $c'_j$  and  $c''_j$  ( $j = 0, \dots, d$ ) such that  $\phi_{C^\bullet}(c_0 \otimes c_1^{-1} \otimes \dots \otimes c_d^{-1}) = x$  and

$$\Phi_\lambda \circ \phi_{C_{(\lambda, \infty)}^\bullet}(c''_0 \otimes (c''_1)^{-1} \otimes \dots \otimes (c''_d)^{-1}) = 1.$$

Then

$$\Psi_\lambda \circ \phi_{C_{[0, \lambda]}^\bullet}(c'_0 \otimes (c'_1)^{-1} \otimes \dots \otimes (c'_d)^{-1}) = \pm x$$

and from (2.11) we get

$$\tau_{\Gamma(\lambda, \infty)} \circ \phi_{C^\bullet(\lambda, \infty)}(c_0'' \otimes (c_1'')^{-1} \otimes \cdots \otimes (c_d'')^{-1}) = \prod_{j=0}^d \text{Det}(\mathcal{B}_{(\lambda, \infty)}^2|_{C_{(\lambda, \infty)}^j})^{(-1)^j j}.$$

Hence, (2.20) is equivalent to (2.21).  $\square$

### 3. THE QUADRATIC FORM ASSOCIATED TO THE SQUARE OF THE ODD SIGNATURE OPERATOR

Let  $E \rightarrow M$  be a complex vector bundle over a closed oriented manifold of *odd* dimension  $d = 2r - 1$  and let  $\nabla$  be a flat connection on  $E$ . Further, let  $\Omega^\bullet(M, E)$  denote the de Rham complex of  $E$ -valued differential forms on  $M$ . For a given Riemannian metric  $g^M$  on  $M$  denote by

$$\Gamma = \Gamma(g^M) : \Omega^\bullet(M, E) \longrightarrow \Omega^\bullet(M, E)$$

the chirality operator (cf. [3, §3]), defined in terms of the Hodge  $*$ -operator by the formula

$$\Gamma \omega := i^r (-1)^{\frac{k(k+1)}{2}} * \omega, \quad \omega \in \Omega^k(M, E). \quad (3.1)$$

The odd signature operator introduced by Atiyah, Patodi, and Singer [1, 2] (see also [20]) is the first order elliptic differential operator  $\mathcal{B} : \Omega^\bullet(M, E) \rightarrow \Omega^\bullet(M, E)$ , given by

$$\mathcal{B} = \mathcal{B}(\nabla, g^M) \stackrel{\text{Def}}{=} \Gamma \nabla + \nabla \Gamma.$$

Note that the operator  $\mathcal{B}$  is elliptic and its leading symbol is self-adjoint with respect to any Hermitian metric on  $E$ . Remark also that  $\mathcal{B}^2$  maps  $\Omega^j(M, E)$  into itself for every  $j = 0, \dots, d$ . We denote by  $(\mathcal{B}^2)_j$  the restriction of  $\mathcal{B}^2$  to  $\Omega^j(M, E)$ .

For an interval  $\mathcal{I} \subset [0, \infty)$  we denote by  $\Omega_{\mathcal{I}}^j(M, E)$  the image of  $\Omega^j(M, E)$  under the spectral projection of  $(\mathcal{B}^2)_j$  corresponding to the eigenvalues whose absolute value lie in  $\mathcal{I}$ . The space  $\Omega_{\mathcal{I}}^j(M, E)$  contains the span of the generalized eigenforms of  $(\mathcal{B}^2)_j$  corresponding to eigenvalues whose absolute value lies in  $\mathcal{I}$  and coincides with this span if the interval  $\mathcal{I}$  is bounded. In particular, since  $\mathcal{B}$  is elliptic, if  $\mathcal{I}$  is bounded, then the dimension of  $\Omega_{\mathcal{I}}^j(M, E)$  is finite. Since  $\mathcal{B}^2$  and  $\nabla$  commute,  $\Omega_{\mathcal{I}}^\bullet(M, E)$  is a subcomplex of the de Rham complex  $\Omega^\bullet(M, E)$ .

For each  $\lambda \geq 0$ , we have

$$\Omega^\bullet(M, E) = \Omega_{[0, \lambda]}^\bullet(M, E) \oplus \Omega_{(\lambda, \infty)}^\bullet(M, E).$$

The complex  $\Omega_{(\lambda, \infty)}^\bullet(M, E)$  is clearly acyclic. Hence, the cohomology  $H_{[0, \lambda]}^\bullet(M, E)$  of the complex  $\Omega_{[0, \lambda]}^\bullet(M, E)$  is naturally isomorphic to the cohomology  $H^\bullet(M, E)$  of  $\Omega^\bullet(M, E)$ . Further, as  $\Gamma$  commutes with  $\mathcal{B}^2$ , it preserves the space  $\Omega_{[0, \lambda]}^\bullet(M, E)$  and the restriction  $\Gamma_{[0, \lambda]}$  of  $\Gamma$  to this space is a chirality operator on  $\Omega_{[0, \lambda]}^\bullet(M, E)$ .

Denote by  $\mathcal{B}_{\mathcal{I}, j}^2$  the restrictions of  $\mathcal{B}^2$  to  $\Omega_{\mathcal{I}}^j(M, E)$ . Let  $\theta \in (0, 2\pi)$  be an Agmon angle for  $\mathcal{B}_{\mathcal{I}}^2$ , cf. [25], and denote by  $\text{Det}_\theta(\mathcal{B}_{(\lambda, \infty), j}^2)$  the  $\zeta$ -regularized determinant of the operator  $\mathcal{B}_{(\lambda, \infty), j}^2$  defined using the Agmon angle  $\theta$ . Since the leading symbol of  $\mathcal{B}_{(\lambda, \infty), j}^2$  is positive definite this determinant is independent of the choice of  $\theta$ .

For any  $0 \leq \lambda \leq \mu < \infty$ , one easily sees that

$$\prod_{j=0}^d \text{Det}_\theta(\mathcal{B}_{(\lambda, \infty), j}^2)^{(-1)^j j} = \left[ \prod_{j=0}^d \text{Det}_\theta(\mathcal{B}_{(\lambda, \mu], j}^2)^{(-1)^j j} \right] \cdot \left[ \prod_{j=0}^d \text{Det}_\theta(\mathcal{B}_{(\mu, \infty), j}^2)^{(-1)^j j} \right] \quad (3.2)$$

For any given  $\lambda \geq 0$ , denote by  $\tau_{\Gamma_{[0,\lambda]}}$  the quadratic form on the determinant line of  $H_{[0,\lambda]}^\bullet(M, E)$  associated to the chirality operator  $\Gamma_{[0,\lambda]}$ , cf. Definition 2.2. In view of (2.20) and (3.2), the product

$$\tau = \tau(\nabla) := \left[ \prod_{j=0}^d \text{Det}(\mathcal{B}_{(\lambda, \infty), j}^2)^{(-1)^{j+1}j} \right] \cdot \tau_{\Gamma_{[0,\lambda]}} \quad (3.3)$$

viewed as a quadratic form on  $\text{Det}(H^\bullet(M, E))$  is independent of the choice of  $\lambda \geq 0$ . It is also independent of the choice of the Agmon angle  $\theta \in (0, 2\pi)$  of  $\mathcal{B}_{(\lambda, \infty)}^2$ .

**Definition 3.1.** *The quadratic form (3.3) on the determinant line of  $H^\bullet(M, E)$  is called the torsion quadratic form.*

**Theorem 3.2.** *The torsion quadratic form  $\tau$  is independent of the Riemannian metric  $g^M$ .*

*Proof.* Suppose that  $g_t^M$ ,  $t \in \mathbb{R}$ , is a smooth family of Riemannian metrics on  $M$  and let  $\tau_t$  denote the torsion quadratic form corresponding to the metric  $g_t^M$ . We need to show that  $\tau_t$  is independent of  $t$ .

Let  $\Gamma_t$  denote the chirality operator corresponding to the metric  $g_t^M$ , cf. (3.1), and let  $\mathcal{B}(t) = \mathcal{B}(\nabla, g_t^M)$  denote the odd signature operator corresponding to  $\Gamma_t$ .

Fix  $t_0 \in \mathbb{R}$  and choose  $\lambda \geq 0$  so that there are no eigenvalues of  $\mathcal{B}(t_0)^2$  whose absolute values are equal to  $\lambda$ . Then there exists  $\delta > 0$  such that the same is true for all  $t \in (t_0 - \delta, t_0 + \delta)$ . In particular, if we denote by  $\Omega_{[0,\lambda],t}^\bullet(M, E)$  the span of the generalized eigenvectors of  $\mathcal{B}(t)^2$  corresponding to eigenvalues with absolute value  $\leq \lambda$ , then  $\dim \Omega_{[0,\lambda],t}^\bullet(M, E)$  is independent of  $t \in (t_0 - \delta, t_0 + \delta)$ .

Let  $\rho_{\Gamma_{t,[0,\lambda]}}$  denote the refined torsion of the pair  $(\Omega_{[0,\lambda],t}^\bullet(M, E), \Gamma_t)$ , cf. Subsection 2.3. As above we shall view  $\rho_{\Gamma_{t,[0,\lambda]}}$  as an element of  $\text{Det}(H^\bullet(M, E))$  via the canonical isomorphism between  $H^\bullet(M, E)$  and  $H_{[0,\lambda]}^\bullet(M, E)$ .

In [8] we fixed a particular square root of  $\prod_{j=0}^d \text{Det}_\theta(\mathcal{B}(t)_{(\lambda, \infty), j}^2)^{(-1)^{j+1}j}$  (In [8] it is denoted by  $e^{\xi_\lambda(t, \theta_0)}$ ). By Lemma 9.2 of [8] the element

$$\rho := \sqrt{\prod_{j=0}^d \text{Det}_\theta(\mathcal{B}(t)_{(\lambda, \infty), j}^2)^{(-1)^{j+1}j}} \cdot \rho_{\Gamma_{t,[0,\lambda]}} \in \text{Det}(H^\bullet(M, E)). \quad (3.4)$$

is independent of  $t \in (t_0 - \delta, t_0 + \delta)$ .

Let  $\tau_{\Gamma_{t,[0,\lambda]}}$  denote the torsion quadratic form of the pair  $(\Omega_{[0,\lambda],t}^\bullet(M, E), \Gamma_t)$ . By (2.7) we have

$$\tau_t(\rho) = \prod_{j=0}^d \text{Det}(\mathcal{B}(t)_{(\lambda, \infty), j}^2)^{(-1)^{j+1}j} \cdot \tau_{\Gamma_{t,[0,\lambda]}}(\rho) = \tau_{\Gamma_{t,[0,\lambda]}}(\rho_{\Gamma_{t,[0,\lambda]}}) = 1, \quad (3.5)$$

where in the latter equality we used (2.7). Thus  $\tau_t(\rho)$  is independent of  $t \in (t_0 - \delta, t_0 + \delta)$ . Since this is true for an arbitrary value of  $t_0$  the theorem is proven.  $\square$

*Remark 3.3.* One can easily give a direct proof of Theorem 3.2, avoiding any references to [8]. One only needs to repeat most of the computations of the proof of Lemma 9.2 of [8]. However, to save space we preferred to use this lemma, rather than repeat its proof.

#### 4. THE RELATIONSHIP WITH BURGHELEA-HALLER AND FARBER-TURAEV TORSIONS

In this section we show that the torsion quadratic form  $\tau$  coincides with the quadratic form defined in [5] and use the results of [5] to establish the relationship between  $\tau$  and the Burghelea-Haller and Farber-Turaev torsions.



**4.1. Relationship with the refined analytic torsion.** Let  $\eta(\nabla) = \eta(\nabla, g^M)$  denote the  $\eta$ -invariant of the restriction of the odd signature operator  $\mathcal{B} = \mathcal{B}(\nabla, g^M)$  to the even forms, see [20], [6, §4], [8, §6.15], or [5, §2.9] for the definition of the  $\eta$ -invariant of a non-self-adjoint operator. Let  $\eta_{\text{trivial}}$  be the  $\eta$ -invariant of trivial line bundle over  $M$ . Let  $\rho_{\text{an}} = \rho_{\text{an}}(\nabla) \in \text{Det}(H^\bullet(M, E))$  denote the refined analytic torsion of  $(E, \nabla)$ , cf. [8, Definition 9.8].

**Proposition 4.2.**  $\tau_\nabla(\rho_{\text{an}}(\nabla)) = e^{-2\pi i (\eta(\nabla) - \text{rank } E \cdot \eta_{\text{trivial}})}.$

It follows that the torsion quadratic form  $\tau$  coincides with the quadratic form defined by equation (1.1) of [5].

*Proof.* Recall that the element  $\rho \in \text{Det}(H^\bullet(M, E))$  is defined in (3.4). From definition of the refined analytic torsion, [8, Definition 9.8], and formulae (9-5) and (10-21) of [8] we conclude that

$$\rho_{\text{an}}(\nabla) = \pm \rho \cdot e^{-\pi i (\eta(\nabla) - \text{rank } E \cdot \eta_{\text{trivial}})}.$$

Hence, the statement of the proposition follows immediately from (3.5).  $\square$

**4.3. Relationship with the Burghlelea-Haller torsion.** Burghlelea and Haller [12, 11] have introduced a refinement of the square of the Ray-Singer torsion for a closed manifold of arbitrary dimension, provided that the complex vector bundle  $E$  admits a non-degenerate complex valued symmetric bilinear form  $b$ . They defined a complex valued quadratic form

$$\tau^{\text{BH}} = \tau_{b, \nabla}^{\text{BH}} \quad (4.1)$$

on the determinant line  $\text{Det}(H^\bullet(M, E))$ , which depends holomorphically on the flat connection  $\nabla$  and is closely related to the square of the Ray-Singer torsion. We refer the reader to [12, 11] for the precise definition of the form  $\tau^{\text{BH}}$  (see also [5, §3] for a short review). Using Proposition 4.2 we now can reformulate Theorem 1.6 of [5] as follows:

**Theorem 4.4.** *Suppose  $M$  is a closed oriented manifold of odd dimension  $d = 2r - 1$  and let  $E$  be a complex vector bundle over  $M$  endowed with a flat connection  $\nabla$ . Assume that there exists a symmetric bilinear form  $b$  on  $E$  so that the quadratic form (4.1) on  $\text{Det}(H^\bullet(M, E))$  is defined. Then  $\tau_{b, \nabla}^{\text{BH}} = \pm \tau_\nabla$ .*

Note that though the Burghlelea-Haller form  $\tau^{\text{BH}}$  is defined only if  $E$  admits a non-degenerate bilinear form  $b$ , the torsion quadratic form  $\tau$  exists without this additional assumption. Therefore,  $\tau$  can be viewed as an extension of  $\tau^{\text{BH}}$  to the case when the bilinear form  $b$  does not exist.

**4.5. Relationship with the Farber-Turaev torsion.** The complex valued combinatorial torsion has been introduced by Turaev [27, 28, 29] and, in a more general context, by Farber and Turaev [18, 19]. The Farber-Turaev torsion depends on the Euler structure  $\varepsilon$  and the orientation  $\mathfrak{o}$  of the determinant line of the cohomology  $H^\bullet(M, \mathbb{R})$  of  $M$ . The set of Euler structures  $\text{Eul}(M)$ , introduced by Turaev, is an affine version of the integer homology  $H_1(M, \mathbb{Z})$  of  $M$ . It has several equivalent descriptions [27, 28, 9, 13]. For our purposes, it is convenient to adopt the definition from Section 6 of [28], where an Euler structure is defined as an equivalence class of nowhere vanishing vector fields on  $M$  – see [28, §5] for the description of the equivalence relation. The Farber-Turaev torsion, depending on  $\varepsilon$ ,  $\mathfrak{o}$ , and  $\nabla$ , is an element of the determinant line  $\text{Det}(H^\bullet(M, E))$ , which we denote by  $\rho_{\varepsilon, \mathfrak{o}}(\nabla)$ .

Suppose  $M$  is a closed oriented odd dimensional manifold. Let  $\varepsilon \in \text{Eul}(M)$  be an Euler structure on  $M$  represented by a non-vanishing vector field  $X$ ,  $\varepsilon = [X]$ . Fix a Riemannian metric  $g^M$  on  $M$  and let

$\Psi(g^M) \in \Omega^{d-1}(TM \setminus \{0\})$  denote the Mathai-Quillen form, [21, §7], [4, pp. 40-44]. Set

$$\alpha_\varepsilon = \alpha_\varepsilon(g^M) := X^* \Psi(g^M) \in \Omega^{d-1}(M).$$

This is a closed differential form, whose cohomology class  $[\alpha_\varepsilon] \in H^{d-1}(M, \mathbb{R})$  is closely related to the integer cohomology class, introduced by Turaev [28, §5.3] and called *the characteristic class*  $c(\varepsilon) \in H_1(M, \mathbb{Z})$  associated to an Euler structure  $\varepsilon$ . More precisely, let  $\text{PD} : H_1(M, \mathbb{Z}) \rightarrow H^{d-1}(M, \mathbb{Z})$  denote the Poincaré isomorphism. For  $h \in H_1(M, \mathbb{Z})$  we denote by  $\text{PD}'(h)$  the image of  $\text{PD}(h)$  in  $H^{d-1}(M, \mathbb{R})$ . Then

$$\text{PD}'(c([X])) = -2[\alpha_\varepsilon] = -2[X^* \Psi(g^M)], \quad (4.2)$$

Burghelea and Haller made a conjecture, [11, Conjecture 5.1], relating the quadratic form  $\tau_{b, \nabla}^{\text{BH}}$  and  $\rho_{\varepsilon, \mathbf{o}}(\nabla)$ , which extends the Bismut-Zhang theorem [4]. In [5, Conjecture 1.12] we extended this conjecture to the case when  $E$  does not admit a non-degenerate symmetric bilinear form. In view of Proposition 4.2 this conjecture can be reformulated as follows.

Following Farber [17], we denote by  $\mathbf{Arg}_\nabla$  the unique cohomology class  $\mathbf{Arg}_\nabla \in H^1(M, \mathbb{C}/\mathbb{Z})$  such that for every closed curve  $\gamma$  in  $M$  we have

$$\det(\text{Mon}_\nabla(\gamma)) = \exp(2\pi i \langle \mathbf{Arg}_\nabla, [\gamma] \rangle),$$

where  $\text{Mon}_\nabla(\gamma)$  denotes the monodromy of the flat connection  $\nabla$  along the curve  $\gamma$  and  $\langle \cdot, \cdot \rangle$  denotes the natural pairing  $H^1(M, \mathbb{C}/\mathbb{Z}) \times H_1(M, \mathbb{Z}) \rightarrow \mathbb{C}/\mathbb{Z}$ .

**Conjecture 4.6.** *Assume that  $(E, \nabla)$  is a flat vector bundle over a closed odd dimensional oriented manifold  $M$ . Then*

$$\tau_\nabla(\rho_{\varepsilon, \mathbf{o}}(\nabla)) = e^{2\pi i \langle \mathbf{Arg}_\nabla, c(\varepsilon) \rangle}. \quad (4.3)$$

The original Burghelea-Haller conjecture was proven independently by Burghelea-Haller [10] and Su-Zhang [26]. Using this result, Theorem 4.4, and formula (1.12) of [5] we obtain the following theorem, which establishes Conjecture 4.6 up to sign in the case when  $E$  admits a non-degenerate bilinear form:

**Theorem 4.7.** *Suppose  $M$  is a closed oriented manifold of odd dimension  $d = 2r - 1$  and let  $E$  be a complex vector bundle over  $M$  endowed with a flat connection  $\nabla$ . Assume that there exists a symmetric bilinear form  $b$  on  $E$ . Then*

$$\tau_\nabla(\rho_{\varepsilon, \mathbf{o}}(\nabla)) = \pm e^{2\pi i \langle \mathbf{Arg}_\nabla, c(\varepsilon) \rangle}. \quad (4.4)$$

Also from Proposition 4.2 and Theorem 1.14 of [5] we obtain the following

**Theorem 4.8.** (i) *Under the same assumptions as in Conjecture 4.6, for each connected component  $\mathcal{C}$  of the set  $\text{Flat}(E)$  of flat connections on  $E$  there exists a constant  $R_\mathcal{C} \in \mathbb{C}$  with  $|R_\mathcal{C}| = 1$ , such that*

$$\tau_\nabla(\rho_{\varepsilon, \mathbf{o}}(\nabla)) = R_\mathcal{C} \cdot e^{2\pi i \langle \mathbf{Arg}_\nabla, c(\varepsilon) \rangle}, \quad \text{for all } \nabla \in \mathcal{C}. \quad (4.5)$$

(ii) *If the connected component  $\mathcal{C}$  contains an acyclic Hermitian connection then  $R_\mathcal{C} = 1$ , i.e.,*

$$\tau_\nabla(\rho_{\varepsilon, \mathbf{o}}(\nabla)) = e^{2\pi i \langle \mathbf{Arg}_\nabla, c(\varepsilon) \rangle}, \quad \text{for all } \nabla \in \mathcal{C}. \quad (4.6)$$

Note that the proof of Theorem 4.8 was obtained in [5] by much softer methods than those used in the proof of the original Burghelea-Haller conjecture [10, 26].

## 5. THE CAPPELL-MILLER TORSION

In this section we first recall the definition of Cappell-Miller torsion

$$T \in \text{Det}(H^\bullet(M, E)) \otimes \text{Det}(H^\bullet(M, E))$$

from [14], then establish its relationship with the torsion form  $\tau$ , and finally, under some additional assumptions, express  $T$  in terms of the Farber-Turaev torsion  $\rho_{\varepsilon, \sigma}$ .

**5.1. The Cappell-Miller torsion of a finite dimensional complex.** Let the complex  $(C^\bullet, \partial)$  and the involution  $\Gamma$  be as in Subsection 2.1. Recall that the element  $\rho_\Gamma \in \text{Det}(H^\bullet(\partial))$  was introduced in (2.6).

In Section 5 of [14] Cappell and Miller introduced a torsion of a class of finite dimensional complexes, which in case of a complex of odd length  $d = 2r - 1$  and in the presence of the involution  $\Gamma$  can be described as

$$T = T_\Gamma := \rho_\Gamma \otimes \rho_\Gamma \in \text{Det}(H^\bullet(\partial)) \otimes \text{Det}(H^\bullet(\partial)). \quad (5.1)$$

The torsion quadratic form  $\tau_\Gamma$  defined in (2.3) can be viewed as an element of

$$\text{Det}(H^\bullet(\partial))^* \otimes \text{Det}(H^\bullet(\partial))^* \simeq \left( \text{Det}(H^\bullet(\partial)) \otimes \text{Det}(H^\bullet(\partial)) \right)^*.$$

It follows from (2.7) that  $\tau_\Gamma$  is the dual of  $T_\Gamma$ , i.e.

$$\tau_\Gamma(T_\Gamma) = 1. \quad (5.2)$$

In particular, if the complex  $(C^\bullet, \partial)$  is acyclic, then  $T$  can be viewed as a complex number via the isomorphism  $\text{Det}(H^\bullet(\partial)) \simeq \mathbb{C}$ , and in this case  $T = 1/\tau$ . It follows now from Lemma 2.7 that if the operator (2.9) is invertible, then

$$T_\Gamma = \prod_{j=0}^d \text{Det}(\mathcal{B}^2|_{C^j})^{(-1)^{j+1}j}. \quad (5.3)$$

*Remark 5.2.* In [14] the element  $T$  is defined in slightly different terms. However, comparing the construction of  $\rho_\Gamma$  with the construction of Section 5 of [14] one immediately sees that our definition coincides with the one of Cappell-Miller up to sign. To see that the signs agree one compares (5.3) with formula (5.43) of [14].

**5.3. The Cappell-Miller torsion of a flat vector bundle.** Let  $E \rightarrow M$  be as in Section 3. Fix a Riemannian metric  $g^M$  on  $M$  and let  $\Gamma$  denote the chirality operator (3.1). We shall use the notation introduced in Section 3. In particular, for each subset interval  $\mathcal{I} \subset [0, \infty)$  we denote by  $\Omega_{\mathcal{I}}^j(M, E)$  the image of  $\Omega^j(M, E)$  under the spectral projection of  $\mathcal{B}^2|_{C^j}$  corresponding to the eigenvalues whose absolute value lie in  $\mathcal{I}$ . Also  $\mathcal{B}_{j, \mathcal{I}}$  denotes the restriction of  $\mathcal{B}$  to  $\Omega_{\mathcal{I}}^j(M, E)$  and  $\Gamma_{\mathcal{I}}$  denotes the restriction of  $\Gamma$  to  $\Omega_{\mathcal{I}}^\bullet(M, E)$ .

Fix  $\lambda > 0$  and let  $T_{\Gamma_{[0, \lambda]}}$  be the Cappell-Miller torsion of the complex  $\Omega_{[0, \lambda]}^j(M, E)$  corresponding to the chirality operator  $\Gamma_{[0, \lambda]}$ . Via the canonical isomorphism  $H_{[0, \lambda]}^\bullet(M, E) \simeq H^\bullet(M, E)$  we can view  $T_{\Gamma_{[0, \lambda]}}$  as an element of  $\text{Det}(H^\bullet(\partial)) \otimes \text{Det}(H^\bullet(\partial))$ .

**Definition 5.4.** Let  $\theta \in (0, 2\pi)$  be an Agmon angle for the operator  $\mathcal{B}_{(\lambda, \infty)}^2$ . The Cappell-Miller torsion  $T_\nabla$  of the flat vector bundle  $(E, \nabla)$  over a closed oriented odd-dimensional manifold  $M$  is the element

$$T_\nabla := \left[ \prod_{j=0}^d \text{Det}_\theta(\mathcal{B}_{(\lambda, \infty), j}^2)^{(-1)^{j+1}j} \right] \cdot T_{\Gamma_{[0, \lambda]}} \in \text{Det}(H^\bullet(\partial)) \otimes \text{Det}(H^\bullet(\partial)). \quad (5.4)$$

It is shown in [14, Theorem 7.3] and also follows from Theorem 5.5 below that  $T_\nabla$  is independent of the choice of  $\lambda$ .

From (5.2), (5.4), and the definition (3.3) of  $\tau_\nabla$  we obtain the following

**Theorem 5.5.**  $\tau_\nabla(T_\nabla) = 1$ .

Hence, Conjecture 4.6 can be reformulated in the form

$$T_\nabla = e^{-2\pi i \langle \mathbf{Arg}_\nabla, c(\varepsilon) \rangle} \cdot \rho_{\varepsilon, \mathfrak{o}}(\nabla) \otimes \rho_{\varepsilon, \mathfrak{o}}(\nabla). \quad (5.5)$$

Let  $E^*$  denote the vector bundle dual to  $E$ . In particular, the fiber  $E_x^*$  of  $E^*$  at a point  $x \in M$  is the dual vector space  $E_x^* = \text{Hom}_{\mathbb{C}}(E_x, \mathbb{C})$ . Let  $\nabla^*$  denote the connection on  $E^*$  dual to  $\nabla$ . Then the direct sum bundle  $E \oplus E^*$  with the connection  $\nabla \oplus \nabla^*$  is unimodular and its fibers have even dimension. Hence, cf., for example, Lemmas 3.2 and 3.3 of [19], the Reidemeister torsion

$$\rho^R(\nabla \oplus \nabla^*) \in \text{Det}(H^\bullet(M, E \oplus E^*)) \simeq \text{Det}(H^\bullet(M, E)) \otimes \text{Det}(H^\bullet(M, E^*)).$$

is well defined and is equal to the Farber-Turaev torsion  $\rho_{\varepsilon, \mathfrak{o}}(\nabla \oplus \nabla^*)$ . In particular,  $\rho_{\varepsilon, \mathfrak{o}}(\nabla \oplus \nabla^*)$  is independent of  $\varepsilon$  and  $\mathfrak{o}$ .

Farber and Turaev, [19, p. 219], introduced the duality operator

$$D : \text{Det}(H^\bullet(M, E)) \rightarrow \text{Det}(H^\bullet(M, E^*)).$$

Using the definition of the Poincaré-Reidemeister scalar product, cf. pages 206 and 219 of [19] and Theorem 9.4 of [19] we obtain

$$\rho_{\varepsilon, \mathfrak{o}}(\nabla) \otimes D(\rho_{\varepsilon, \mathfrak{o}}(\nabla)) = (-1)^z e^{2\pi i \langle \mathbf{Arg}_\nabla, c(\varepsilon) \rangle} \cdot \rho^R(\nabla \oplus \nabla^*),$$

where  $z \in \mathbb{N}$  is defined in formula (6.5) of [19]. Hence, (5.5) is equivalent to the following conjecture, originally made by Cappell and Miller [14]:

**Conjecture 5.6 (Cappell-Miller).** *Assume that  $(E, \nabla)$  is a flat vector bundle over a closed odd dimensional oriented manifold  $M$ . Then the Cappell-Miller torsion is related to the Reidemeister torsion by the equation*

$$(1 \otimes D)T_\nabla = (-1)^z \rho^R(\nabla \oplus \nabla^*), \quad (5.6)$$

where  $z \in \mathbb{N}$  is defined in formula (6.5) of [19]<sup>2</sup>

Theorems 4.7 and 4.8 give a partial solution of this conjecture. In particular, Theorem 4.8 says that Conjecture 5.6 holds up to the factor  $R_C$  and holds exactly in the case when  $\nabla$  belongs to a connected component of the space  $\text{Flat}(E)$  which contains an acyclic Hermitian connection. Theorem 4.7 states that Conjecture 5.6 holds up to sign if  $E$  admits a non-degenerate bilinear form  $b$ .

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